A Cross-Age Study of Students' Understanding of Fractals

Um Estudo sobre o Modo como Alunos Compreendem Fractais

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Abstract

The purpose of this study is to examine how students understand fractals depending on age. Students' understandings were examined in four dimensions: defining fractals, determining fractals, finding fractal patterns rules and mathematical operations with fractals. The study was conducted with 187 students (grades 8, 9, 10) by using a two-tier test consisting of nine questions prepared based on the literature and Turkish mathematics and geometry curriculums. The findings showed that in all grades, students may have misunderstandings and lack of knowledge about fractals. Moreover, students can identify and determine the fractals, but when the grade level increased, this success decreases. Although students were able to intuitively determine a shape as fractal or not, they had some problems in finding pattern rules and formulizing them.

Keywords: Fractal Geometry. Students' Understandings. Cross-age Study.

Resumo

O objetivo deste estudo é analisar como os alunos compreendem fractais, dependendo da idade. Entendimentos dos alunos foram examinados em quatro dimensões: a definição de fractais, determinando fractais, encontrando padrões fractais e operações matemáticas com fractais. O estudo foi conduzido com 187 estudantes do Ensino Fundamental e Médio (grau 8, 9, 10) usando um teste de dois níveis composto de nove questões

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elaboradas por meio da literatura nos currículos turcos em matemática e geometria. Os resultados mostraram que em todos os graus, os alunos podem ter falta e conhecimentos equivocados sobre fractais. Além disso, os estudantes podem identificar e determinar os fractais, mas quando o nível de ensino aumenta, a identificação e a determinação dos fractais diminui. Embora, os alunos pudessem determinar intuitivamente uma forma como fractal ou não, eles tiveram alguns problemas em encontrar regras e padrões na formulação dos fractais.

Palavras-chave: Geometria Fractal. Entendimentos dos Alunos. Faixa-etária do Estudo.

1 Introduction

Fractal Geometry, which differs from traditional geometry, is a relatively new and important area of Mathematics and Mathematics Education. In recent years, Fractal Geometry appears in many mathematics curricula with the reforms in Mathematics Education. Although there is a desire to include fractals in the mathematics curriculum, there is little knowledge of how students would learn the content and react to them. Revealing students' knowledge and understandings about fractals in the current mathematics curriculum is important for determining possible difficulties of the subject and overcoming the difficulties. Moreover, revealing knowledge and understandings about fractals of students at differing grade levels may provide clues about kinds of changes in these students' knowledge and understandings. It is known that at some grades, students experience conceptual difficulties regarding fractals (BOWERS, 1991; LANGILLE, 1996; KOMOREK et al., 2001; KARAKUŞ, 2011).

1.1 Students' understanding of fractals

Studies about teaching and learning Fractal Geometry have been divided into two subsections: theory and practices.

In the first section of studies, activities are frequently developed about teaching and learning fractals for teachers who are applying them in their classroom (THOMAS, 1989; GOLDENBERG, 1991; COES, 1993; NAYLOR, 1999; BARTON, 2003; DEVANEY, 2004; FRABONI; MOLLER, 2008). Other studies describe the experiences of teachers who have actually experimented with fractals in their classrooms (SIMMT; DAVIS, 1998; VACC, 1999; LORNELL; WESTERBERG, 1999; IOVINELLI, 2000; BOLTE, 2002; FORD, 2004). However, there is little knowledge of how students would learn these subjects, what kind of knowledge and understandings they have and what kind of difficulties they have. For example, Bowers (1991) conducted a study about teaching fractals with three 12th grade students. In this study, Bowers examined students' difficulties in forming the meaning of fractal concept, their understandings related to fractal concept and in which environments this concept can be taught. Bowers (1991) determined that students have difficulties in three subjects in learning of fractals. The first difficulty is learning fractal dimension, the second one is determining the scaling factor in the self-similar parts and the third is construction of a fractal which is not obvious. Bowers especially stated that students' difficulties in determining the differences between the process of building the object and the object itself, and difficulties in understanding the relation between iterative process and the definition of the fractal, presented obstacles to the students understanding of fractals. Also, Bowers claimed that another reason for students' difficulties in understanding fractals is an epistemological obstacle about fractals. Similarly, Langille (1996) conducted a study about integration of fractal geometry into the 12th grade mathematics curriculum. He determined that students have difficulties in understanding dimension, self-similarity dimension and in determining characteristics of fractals. Komorek et al. (2001) investigated that teaching fractal geometry and chaos theory in science education is suitable or not for aged between 15-17 students. The results of the study revealed that students can determine an objects' self-similarity intuitively, but they have difficulties in defining self-similarity mathematically and in determining the magnification factor (scaling ratio). Karakuş (2011) determined that pre-service teachers can generally decide whether a given shape is a fractal or not, but they have difficulties in finding patterns about perimeter and area of fractals and learning of fractal dimension. He stated that pre-service teachers have difficulties in determining self-similar parts number and magnification factor in calculating the fractal dimension of an object. Moreover, self-similarity of natural objects is another obstacle for pre-service teachers in forming self-similarity concepts in their minds. The most important reason for this, according to the author, was the definition used in the explanation of self-similarity concept. Because the definition of self-similarity based on the comparison of part and whole of a fractal was inadequate.

1.2 The purpose of the study

As can be seen from the literature, there is no study concerning what students know about fractals, or how students' understanding about fractals changes with respect to grade level. Therefore the present work seeks to investigate this gap in the literature and focused on how students understand fractals. In this context, the study aims to examine how students understand the fractals depending on their age.

2 Method

2.1 The context of the study

The schooling system in Turkey consists of three main components: basic education (elementary schools, age 6-14; 8 years), which is compulsory; secondary education (lycees or senior high schools, age 14-17; 4 years), and higher education (colleges and universities) (ÇALIK, 2005). All schools throughout the country must use the same curricula, which is prepared and implemented by the National Ministry of Education (MEB) (TURKEY, 2005).

In Turkish educational system, teaching of fractals begins with an introduction to fractals at the age of 13-14 years in Grade 8. The Grade 8 mathematics curriculum includes a goal about fractals "To build patterns from line, polygon and circle models, to draw them and to determine fractals from these patterns" (TURKEY, 2008, p. 316). The goal was prepared in terms of building fractal patterns by using figures in Euclid Geometry or deciding whether given patterns are fractals or not. In that grade, fractals are taught by drawing activities and finding fractal patterns. Moreover, there are natural fractal examples such as ferns or tree branches in the Grade 8 mathematics curriculum. Fractal activities related to finding similar patterns such as triangle numbers, edge numbers and corner numbers in the fractal generation steps are taught to students aged between 14-15 years (Grade 9). However, there is not a direct goal related to teaching fractals in the Grade 9 geometry curriculum. Finding fractal patterns and mathematical operations with them are taught to students aged 16-17 years (Grade 10). There are two goals about fractals in the Grade 10 geometry curriculum. The first goal is "To build fractals with segments, to explain them and to compute the length of the fractal in a particular step" (TURKEY, 2010, p. 153) and the second is "To build fractals with triangles, to explain them and to compute the area of fractal image in a particular step" (TURKEY, 2010, p. 157).

In conclusion, in Turkish mathematics and geometry curriculums, fractals are studied as follows; recognition of fractals in Grade 8, finding patterns and mathematical operations with them in Grades 9 and 10.

2.2 Instrument and data collection procedure

To examine students' understandings, according to their grade levels and comprehension, cross-age and longitudinal studies are generally used (ABRAHAM et al., 1994; COHEN et al., 2007). However, Abraham et al. (1994) have implied that when time is limited, cross-age studies are more suitable than longitudinal studies. Also, a number of researchers have reported successful cross-age studies (e.g., SHARPLEY et al., 1983; TOPPING et al., 2003; KARATAŞ et al., 2011). Therefore, in this study, a cross-age study was undertaken for grades 8-10.

A paper-pencil test comprising nine two-tier questions was prepared (Appendix). The first part of the questions consists of multiple choices and the other part of the questions requires students to explain their reason why they selected this choice. These kinds of two-tier questions are useful because they identify alternative conceptions along with the underlying reasons (ÇALIK, 2005). The test contains four main categories. These categories and their aims are summarized in Frame 1.

Categories	Items	Content of the questions	Aims
Definition of fractals	1, 3	We asked the students the definition of fractals and the definition of self-similarity which is the most important characteristic of a fractal.	To determine students' knowledge about the definition of fractals and their features.
Determining fractals	2, 4, 6	We asked the students to determine which shape is fractal or not	To determine whether students recognize a fractal shape or not
Finding fractal patterns rule	5, 7	We asked the students to find the generator of a fractal and the iteration rule of a fractal.	To find the iteration rule of a fractal and to determine the generator of a fractal
Mathematical operations with fractal patterns	8, 9	We asked the students to calculate the number of parts which are added fractals and to find a pattern in the area of fractal.	To carry out some mathematical operations with fractals.

Frame 1 – Categories of fractal test
Source: Research data

Before administering the final form of the test, a group consisting of mathematics educators and mathematics teachers checked the validity of test items and agreed that the items were valid and appropriate to measure students' understanding of the fractals.

2.3 Sample

The sample consisted of 187 students in different grades that ranged from Grade 8 (13-14 years) to Grade 10 (16-17 years). There were 72 students from Grade 8, 69 students from Grade 9 and 46 students from Grade 10. The sample was selected at random from one elementary and two secondary schools in the city of Trabzon in Turkey. The concepts were initially introduced to students in Grade 8 and then the concepts were introduced at a more advanced level to the students in Grades 9 and 10.

2.4 Data analysis

The following categorizations were used to analyze students' test responses: a correct choice with the correct reason (6 points), a correct choice with partially correct reason (5 points), incorrect (or no selected) choice but correct reason (4 points), incorrect choice with partially correct reason (3 points), correct choice but inadequate/weak correct reason (2 points), correct choice (1 point), incorrect choice/no answer (0 points). The use of these classification criteria for student responses provides an opportunity to compare students´ understanding to researchers (ÇALIK, 2005). Some researchers (e.g. ÇALIK, 2005; KARATAŞ et al., 2011) used similar categories for examining students' understandings. Differences among group test scores were investigated by traditional statistical means using parametric tests: one-way ANOVA and Tukey.

3 Results

The analyses and interpretations of the scores obtained by the students on the fractal test will be presented in this section. Students' answers were categorized and analyzed using SPSS software. Table 1 shows the results regarding the scores obtained by the students on the fractal test.

Table 1 – The means and standard deviations of student scores on the fractal test

Categories	Grades	N	M	SD
Definition of fractals	8	72	4,1528	3,66813
	9	69	3,8116	3,37933
	10	46	3,6739	3,41925
Determining fractals	8	72	8,0139	5,00280
	9	69	7,3333	4,89498
	10	46	5,6304	4,21299
Finding fractal patterns rule	8	72	2,1111	2,52623
	9	69	2,2464	2,73542
	10	46	1,6304	2,18438
Mathematical operations with fractal patterns	8	72	1,5694	2,16779
	9	69	1,9420	2,36944
	10	46	1,2609	1,63890
Total success of the fractal test	8	72	15,8472	9,95010
	9	69	15,3333	9,66143
	10	46	12,1957	8,15848

Source: Research data

When the students' scores are examined, it can be seen that while the performances of the 8th grade students were highest in the categories of definition of fractals, determining fractals and the total success of fractal test (M=4,15, SD=3,67; M=8,01, SD=5,00; M=15,85, SD=9,95, respectively), the 9th grade students scored highest in the categories of finding fractal patterns' rule and mathematical operations with fractal patterns (M=2,25, SD=2,75; M=1,94, SD=2,37, respectively). However, the 10th grade students had the lowest performance in all categories.

Students' answers for item 1 and item 3 on the fractal test form the category of definition of fractals. In that category, maximum mean score should be 12, but as can be seen in Table 1, grades' mean scores were lower than maximum mean score (respectively M= 4,15 (%34,6); M=3,81 (%32,8); M= 3,67 (%30,6)). This result shows that students have misunderstandings about the definition of fractals.

Similarly, students' answers for item 2, item 4 and item 6 on the fractal test form the category of determining fractals. In that category, maximum mean score should be 18, but as shown in Table 1, grades' mean scores were lower than maximum mean score (M= 8,01 (%44,5); M=7,33 (%40,7); M= 5,63 (%31,3), respectively). This result shows that students can identify and determine the fractals, but when the grade level increased, this success decreased.

Students' answers for item 5 and item 7 on the fractal test form the category of finding fractal pattern rules. In that category, maximum mean score should be 12, but as shown in Table 1, grades' mean scores were lower than maximum mean score (M= 2,11 (%17,6); M=2,25 (%18,8); M= 1,63 (%13,6), respectively). This result shows that students have difficulty identifying the formation rule of fractal patterns.

Students' answers for item 8 and item 9 on the fractal test form the category of mathematical operations with fractal patterns. In that category, maximum mean score should be 12, but as can be seen in Table 1, grades' mean scores were lower than maximum mean score (M= 1,57 (%13,1); M=1,94 (%16,2); M= 1,26 (%10,5), respectively). This result shows that students have difficulty in carrying out mathematical operations with fractal patterns.

Total success of the fractal test, maximum mean score should be 54, but as shown in Table 1, grades' mean scores were lower (M= 15,85 (%29,4); M=15,33 (%28,4); M=12,20 (%22,59), respectively). This result shows that students' understandings about fractals are low in all grades according to the fractal test.

The one way ANOVA – one way analysis of variance – was applied in order to compare fractal test scores for each grade. The result of the one way ANOVA test is shown in Table 2.

Table 2 – One-way ANOVA results of the students related to the fractal test

Categories		Sum of Squares	df	Mean Square	F	Sig.
Definition of fractals	Between Groups	7,476	2	3,738	0,305	0,738
	Within Groups	2257,979	184	12,272		
	Total	2265,455	186			
Determining fractals	Between Groups	162,140	2	81,070	3,547	,031
	Within Groups	4205,037	184	22,853		
	Total	4367,176	186			
Finding fractal patterns rule	Between Groups	11,018	2	5,509	0,861	0,424
	Within Groups	1176,640	184	6,395		
	Total	1187,658	186			
Mathematical operations with fractal patterns	Between Groups	13,250	2	6,625	1,458	0,235
	Within Groups	836,290	184	4,545		
	Total	849,540	186			
Total success of the fractal test	Between Groups	410,279	2	205,140	2,306	0,103
	Within Groups	16371,892	184	88,978		
	Total	16782,171	186			

($p<0.05$).

Source: Research data

As shown in Table 2, there was no statistically significant difference found in the category of definition of fractals (F(2,184) = 0,305; p = 0,738x), finding fractal patterns' rule (F (2,184) = 0,861, p = 0,424), mathematical operations with fractal patterns (F(2,184) = 1,458; p = 0,235)) and total success of the fractal test (F(2,184) = 2,306, p = 0,103). However, a significant difference was found in the category of determining fractals (F(2,184) = 3,547, p = 0,031). Table 3 shows the results of the Tukey test performed according to the scores of students related to the category of determining fractals.

Table 3 – Results of the Tukey test related to the category of determining fractals

Grades	Mean differences	Std. Error	Sig.
8-9	0,68056	0,80537	0,676
8-10	2,38345*	0,90234	0,024
9-10	1,70290	0,90996	0,150

(p<0.05).

Source: Research data

As a result of the Tukey test performed in order to determine the source of significant differentiation observed, 8th graders were found to be more successful than 10th graders (p<0.05). Moreover, there was no statistically significant difference found between grade 8 and grade 9; grade 9 and grade 10 (p<0.05).

4 Discussion and conclusion

The result of the one-way ANOVA test showed no statistically significant differences between the total successes of the grades, but looking at the total scores of students in all grades, the grade 10 students were found to be considerably less successful than the other students and the grade 8 students were the most successful. We expected that the success of the grades would increase from grade 8 to grade 10. The reason may be the insufficiency of teachers with respect to teaching fractals. Teachers in Turkey have not attended any in-service courses about fractals (BAKI et al., 2008; KARAKUŞ, 2011). Moreover, teaching fractals has continued in grade 8 for three years, in grade 9 for two years and in grade 10 for a year. So, the subject is also new for students. Furthermore, the inadequacy of textbooks may be another reason. Karakuş and Baki (2011) argue that there are many mistakes in the textbooks for teaching fractals in Turkey. Despite the success of the grade 8 students, the total success of all the grades was very low. This shows that in all grades, students may have lack of knowledge and misunderstanding about fractals.

In the category of fractal definition, despite not finding statistically significant differences among grades, according to the scores, grade 8 students define fractal more correctly than the students in other grades. The reason for this may be that there is a definition of fractal only in grade 8 textbooks. In grade 9 and grade 10, there is no definition for fractals. However, the total scores of all grades with respect to the category of fractal definition were low. One of the reasons of this low average maybe that the definition of a fractal is not exactly clear as Bowers (1991) stated, since students may not understand in which step fractal is formed when they were transforming a familiar geometric shape into another shape by iterating in an order. Another reason may be the curriculum, as features of fractals such as self-similarity and iteration were not focused on (KARAKUŞ; BAKI, 2011).

In the category of determining fractals, statistically significant differences were found between grades 8 and 10, with the 8th graders performing better. Grade 8 students can identify and determine the fractals, but when the grade level increases, this success decreases. The reason for that can be that in the grade 8 curriculum, more activities for determining and building fractals were included. It can be said that the current mathematics and geometry curricula have an effect on this. Furthermore, looking at the total scores of the grades, grade 8 and grade 9 students generally determined fractals successfully. This situation shows that students can recognize the fractal shapes intuitively as stated in the studies of Karakuş (2011) and Komorek et al. (2001).

In the category of finding the fractal patterns rule, despite not finding statistically significant differences among grades, according to the scores, grade 9 was more successful than others. The reason of this can be that in the grade 9 and grade 10 curricula, more activities for finding fractal patterns and mathematical operations with them were included. It is determined that the total scores for all grades were low. This shows that students have difficulty in determining the pattern of shape. Previous studies on students (THRELFALL, 1999; STACEY, 1989; ORTON; ORTON, 1999; ZAZKIS; LILJEDAHL, 2002) report same difficulties about finding repeated patterns. Yet, it is interesting the scores of grade 10 were lower than those of grade 8.

In the category of mathematical operations with fractal patterns, despite not finding statistically significant differences among grades, it can be said that grade 9 students were more successful than others. However, the total scores of the grades were the lowest in all categories of the test. This shows that students in all grades have many difficulties in carrying out mathematical operation with fractals. The reason for that can be having difficulties in finding patterns and generalizing them. Previous studies on mathematical operation with patterns (STACEY, 1989; ENGLISH; WARREN, 1998; ORTON; ORTON, 1999; FEIFE, 2005) report same difficulties. Despite not including mathematical activities about fractals in grade 8, these kinds of activities can be found in grade 9 and grade 10. Yet, it is interesting that the success of grade 10 was the lowest.

5 Educational implications

The results of this study showed that students generally determine fractals intuitively. However, there are also common problems with respect to the definition of fractals, finding the rules for patterns and mathematical operations with fractals in all grades. For this reason, some changes to the content of the courses should be made. The definition of fractals can be included in each grade by expanding the content of the definition. Furthermore, one of the most important characteristics of a fractal, self-similarity, can be given as intuitively as well as formally in the textbooks. Thus, the definition of fractal may be more meaningful for students. Also, students have difficulty in finding the rule for fractals patterns. This finding shows that they have deficiencies in understanding how a fractal is built. Therefore, having a formal way to form a fractal, such as initiator-generator-iteration, may be effective in overcoming this difficulty. We think paying attention especially to the generator, which provides the formation of fractal, may support success in both defining and understanding the rule and applying the rule in the iteration steps. The results show that carrying out mathematical operations with fractals was the most problematic subject in all grades. More in-depth studies on this subject can contribute to determining students' proficiencies about finding patterns and formulating them.

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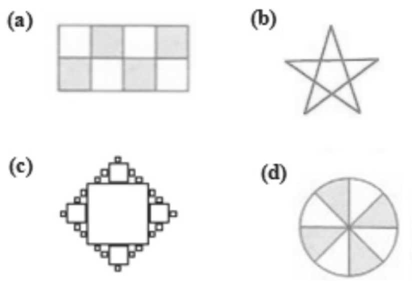
Appendix
Two-Tier Test about Fractals

1. A fractal is a result of
(a) iterating shapes which have same size
(b) iterating shapes which have similar and same size
(c) iterating shapes which have different size
(d) iterating shapes which have similar, but different size

Because……………………………….

2. Which of the following is a fractal?

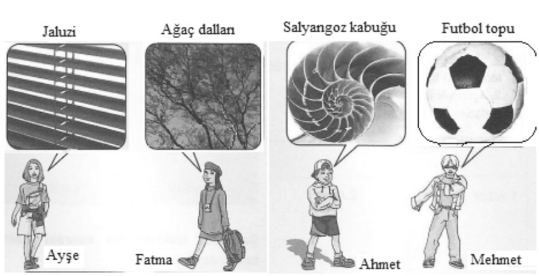


Because……………………………….

3.
(a) Whole parts of a fractal have same size and they are similar.
(b) Many parts of a fractal have same size, but they do not have to be similar
(c) When a part of a fractal is magnified, it is similar to the whole shape
(d) All shapes in any part of a fractal have same size.

Because……………………………….

4. Ayşe, Fatma, Ahmet and Mehmet collect fractal objects for their project.



Which objects collected by students are shown as an example for fractals?
(a) Ayşe ve Fatma
(b) Ayşe ve Ahmet
(c) Fatma ve Ahmet
(d) Ahmet ve Mehmet

Because…………………………………

5. Examine the following pattern

Initial shape first iteration second interation

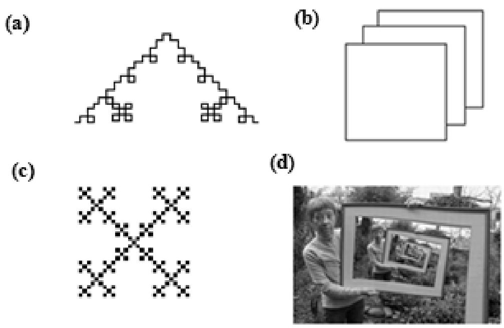
This pattern is a

(a) Fractal (b) Not a fractal

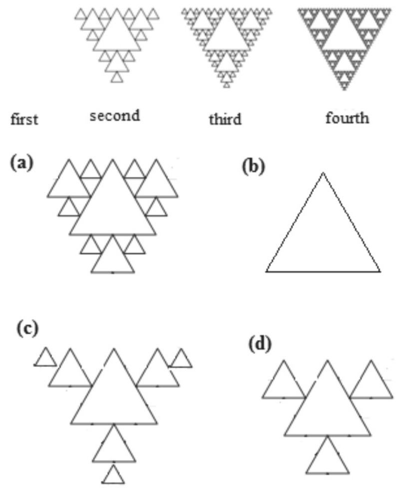
Because…………………………………

6. Which of the following is not a fractal?

Because ………………………………………

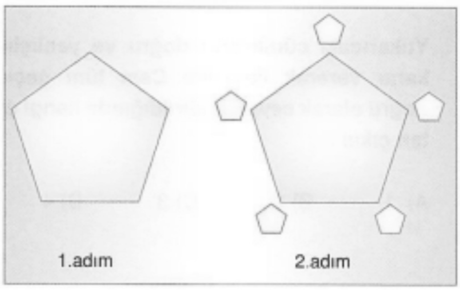


7. Which one is the first iteration of the given pattern?



Because……………………………….

8.



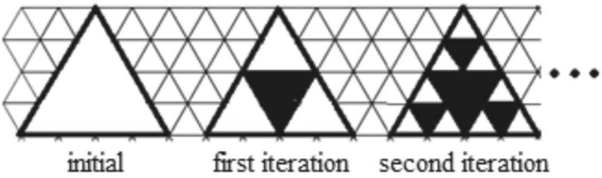
Mehmet wants to calculate how many pentagons are there in the fourth iteration and he has found the answer as 66 by following the steps:
First step: 1
Second step: 1+5
Third step: 1+5+15
Fourth step: 1+5+15+45

In which step did he make his first mistake?

(a) 1 (b) 2 (c) 3 (d) 4

Because……………………………….

9.



In the first iteration of the Sierpinski gasket, the area of the colored triangle is 1 cm^2. In this case, in the 8th iteration step, what is the area of one of the smallest colored triangle?

(a) $\frac{1}{4^2}$ **(b)** $\frac{1}{4^4}$ **(c)** $\frac{1}{2^9}$ **(d)** $\frac{1}{2^{14}}$

Because……………………………….